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A New Systematic Formalism for Similarity Analysis

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SUMMARY

A new systematic formalism is presented for similarity analysis—i.e., for reducing the number of independent variables in systems consisting of partial differential equations and a set of auxiliary conditions. The formalism is a specialization of previous group theory techniques developed by the authors. Concurrent with its presentation, and to illustrate that it is particularly well suited for practical similarity analyses, the formalism is applied to certain three-dimensional incompressible boundary layer flows; and a variety of results are elicited which exhibit somewhat greater generality than any previously reported for the class of flows.

1. Introduction

A new systematic formalism is presented for reducing the number of independent variables in systems which consist, in general, of a set of partial differential equations and a set of auxiliary conditions (such as boundary and/or initial conditions). In engineering, such procedures are customarily termed *similarity analyses*. The formalism is a significant simplification of general group theory techniques developed by the present authors, [1], based upon elementary group theory and upon earlier methods due to Birkhoff [2], Michal [3] and his co-worker Morgan [4]. The principal advantage of the procedure reported here over that presented in [1] lies in the relative ease and rapidity with which the formalism may be applied, and which makes it somewhat more suitable for *practical* similarity analyses. The key to the simplified procedure is this: Rather than initiate an analysis with a very general class of groups, as in [1], every analysis begins with the special, though yet rather general, class of the form (3.1). As a result, many of the manipulations become easier.

The systematic formalism to be introduced here, as well as the general techniques of [1], represent significant advances over previously reported group methods inasmuch as they are *deductive*: Specifically, (*i*) beginning with a general class of groups, an appropriate group (or class of groups) is deduced, (*ii*) the deduction procedure explicitly considers the auxiliary conditions as well as the differential equations, and (*iii*) sets of absolute invariants for the groups—the similarity variables—are systematically derived.

The formalism is explained by application to a problem in boundary layer flow: A system involving *partial* differential equations in *three* independent variables, (2.1)–(2.6), is examined for transformations of variables which enable it to be reduced to a system involving only one independent variable.

2. Illustrative Problem

The systematic formalism has application to a wide range of physical problems. However, to explain it, perhaps the best illustration is a familiar one taken from boundary layer theory: Consider the problem of incompressible laminar boundary layer flow over a flat plate, represented in rectangular coordinates by

TABLE 1

Case	$U(x, z)$	$W(x, z)$	$\eta(x, y, z)$	$\frac{C_1}{U_0}$	$\frac{C_2}{W_0}$	$\frac{C_3}{U_0}$	$\frac{C_4}{W_0}$	$\frac{C_5}{U_0}$	$\frac{C_6}{W_0}$
1	$U_0[x+Az+B]^{2\mu+1}$	$W_0[x+Az+B]^{2\mu+1}$	$y[x+Az+B]^{\mu}$	μ	$A\mu$	$[2\mu+1]$	$A[2\mu+1]$	$A[2\mu+1]$	$[2\mu+1]$
2	$U_0 \exp(2\mu x) \exp(2mz)$	$W_0 \exp(2\mu x) \exp(2mz)$	$y \exp(\mu x) \exp(mz)$	μ	m	2μ	$2m$	$2m$	2μ
3	$U_0[x+A]^{2\mu+1}[z+B]^{2m}$	$W_0[x+A]^{2\mu}[z+B]^{2m+1}$	$y[x+A]^{\mu}[z+B]^{m}$	μ	m	$[2\mu+1]$	$[2m+1]$	$2m$	2μ
4	$U_0[z+B]^{2m} \exp(2\mu x)$	$W_0[z+B]^{2m+1} \exp(2\mu x)$	$y[z+B]^{m} \exp(\mu x)$	μ	m	2μ	$[2m+1]$	$2m$	2μ
5	$U_0[x+A]^{2\mu+1} \exp(2mz)$	$[x+A]^{2\mu} \exp(2mz)$	$y[x+A]^{\mu} \exp(mz)$	μ	m	$[2\mu+1]$	$2m$	$2m$	2μ
6	$U_0[x+A]^{2\mu+1}$	$W_0[x+A]^{\sigma}$	$y[x+A]^{\mu}$	μ	0	$[2\mu+1]$	0	0	σ
7	$U_0[z+B]^{\sigma}$	$W_0[z+B]^{2m+1}$	$y[z+B]^{m}$	0	m	0	$[2m+1]$	σ	0
8	$U_0 \exp(2\mu x)$	$W_0 \exp(\sigma x)$	$y \exp(\mu x)$	μ	0	2μ	0	0	σ
9	$U_0 \exp(\sigma z)$	$W_0 \exp(2mz)$	$y \exp(mz)$	0	m	0	$2m$	σ	0

$$u_x + v_y + w_z = 0 \tag{2.1}$$

$$uu_x + vu_y + wu_z - \nu u_{yy} - UU_x - WU_z = 0 \tag{2.2}$$

$$uw_x + vw_y + ww_z - \nu w_{yy} - UW_x - WW_z = 0 \tag{2.3}$$

$$u(x, 0, z) = v(x, 0, z) = w(x, 0, z) = 0 \tag{2.4}$$

$$\lim_{y \to \infty} u = U(x, z) \tag{2.5}$$

$$\lim_{y \to \infty} w = W(x, z), \tag{2.6}$$

wherein $\nu(\nu > 0)$ represents the constant kinematic viscosity of the fluid; $u(x, y, z)$ symbolizes the velocity component in the x-direction, and $U(x, z)$ $(U > 0)$ is its limit as the normal distance from the plate surface approaches infinity; etc.

The *objective* is to apply the systematic formalism to develop transformations of variables which enable the given *representation*, (2.1)–(2.6), involving the three independent variables x, y, z to be reduced to a representation in one independent variable—to a *similarity representation*. Several different cases can be deduced *concurrently*. These cases are distinguished from one another by the functions $U(x, z)$ and $W(x, z)$; the special form of these functions for each case arises naturally in the simultaneous development of all the cases. In particular, nine distinct cases exist which reduce the number of independent variables from three (x, y, z) to *one* $(\eta(x, y, z))$; see Table 1. The nine cases include as *special cases* results presented earlier in [5]. The more general results obtained here, and especially the fact that all the results evolve naturally in a single analysis, testifies for the approach being presented.

For each case in Table 1 a similarity representation for (2.1)–(2.6) is obtained by means of the transformation of variables $\{u = UF_1(\eta),\ v = F_2(\eta)/y,\ w = WF_3(\eta)\}$ and elementary chain rule operations. Thus, as may be readily verified, for each case in Table 1 (2.1)–(2.6) may be expressed respectively as

$$C_3\eta^2 F_1 + C_1\eta^3 \frac{dF_1}{d\eta} + C_4\eta^2 F_3 + C_2\eta^3 \frac{dF_3}{d\eta} + \eta\frac{dF_2}{d\eta} - F_2 = 0 \tag{2.7}$$

$$C_3[F_1^2 - 1] + C_5[F_1F_3 - 1] + [C_1F_1 + C_2F_3]\eta\frac{dF_1}{d\eta} + \frac{1}{\eta}F_2\frac{dF_1}{d\eta} - \nu\frac{d^2F_1}{d\eta^2} = 0 \tag{2.8}$$

$$C_4[F_3^2 - 1] + C_6[F_1F_3 - 1] + [C_1F_1 + C_2F_3]\eta\frac{dF_3}{d\eta} + \frac{1}{\eta}F_2\frac{dF_3}{d\eta} - \nu\frac{d^2F_3}{d\eta^2} = 0 \tag{2.9}$$

$$\text{as } \eta \to 0: \quad F_1 \to 0, \quad [F_2/\eta] \to 0, \quad F_3 \to 0 \tag{2.10}$$

$$\text{as } \eta \to \infty: \quad F_1 \to 1, \quad F_3 \to 1 \tag{2.11}$$

The manner in which the transformations of variables underlying (2.7)–(2.11) are established by application of the systematic formalism will now be presented. However, inasmuch as a detailed development may be found in [6], the discussion will center principally upon the major points of the formalism. While the present discussion proceeds within the context of the specific example (2.1)–(2.6), the technique is equally applicable to other problems, simply by following the same steps.

3. The Group of Transformations

An elemental feature of any of the group approaches to similarity analysis is the *group of transformations*; (see [7, pp. 10–16], for an introduction to the mathematical group as applied to similarity analysis). It has been typical of previous group approaches to initiate an analysis with a *particular* simple form of group. The manner in which such a group is to be selected has been vaguely explained; but previous experience generally has played a major role in the selection process; (see [7, p. 12]). As a result of that approach, appropriate groups not conforming to custom can be overlooked, and hence so would any conclusions which would follow therefrom. Too, a particular assumed special group need not necessarily lead to fruitful results.

It was in recognition of the difficulties and uncertainties inherent in establishing a suitable group that the authors have proposed, [1], an effective procedure for *deducing* appropriate groups, beginning with a very general class of groups. Experience with the procedure has indicated, however, that for an extensive realm of problems of practical interest the resultant groups satisfy a somewhat more special, though yet rather general, form. Furthermore, the analysis is significantly shortened if initiated with this somewhat special class—a class which, very significantly, includes as a member virtually every one of the simple groups that has *previously* been reported to have utility for similarity analysis.

Thus, in the notation of the given representation, (2.1)–(2.6), the present analysis is initiated with a class C_G of two-parameter transformation groups with the form

$$G: \begin{cases} S: \begin{cases} \bar{x} = C^x(a_1, a_2)x + K^x(a_1, a_2) \\ \bar{y} = C^y(a_1, a_2)y + K^y(a_1, a_2) \\ \bar{z} = C^z(a_1, a_2)z + K^z(a_1, a_2) \end{cases} \\ \bar{u} = C^u(a_1, a_2)u + K^u(a_1, a_2), \quad \bar{U} = C^U(a_1, a_2)U + K^U(a_1, a_2) \\ \bar{v} = C^v(a_1, a_2)v + K^v(a_1, a_2) \\ \bar{w} = C^w(a_1, a_2)w + K^w(a_1, a_2), \quad \bar{W} = C^W(a_1, a_2)W + K^W(a_1, a_2) \end{cases} \tag{3.1}$$

where the real-valued C's and K's are at least differentiable in each real argument, *but are otherwise unspecified*. The a's are termed *parameters*.

For the purpose of the formalism being presented it is necessary that the class of groups C_G have *complete sets* of differentiable *absolute invariants* with a certain form. [A function $g(x, y, z, u, v, w, U, W)$ is said to be an absolute invariant for a group G provided that under the transformations of G: $g(\bar{x}, \bar{y}, \bar{z}, \ldots) = g(x, y, z, \ldots)$. And a set of absolute invariants for G is said to be *complete* if there is no absolute invariant of G that is functionally independent of those in the set.] By definition, C_G is to be comprised of all two-parameter groups satisfying (3.1) which possess complete sets of differentiable absolute invariants with the form

$$\{\eta(x, y, z),\ g_\delta(x, y, z, u, v, w, U, W)\} \qquad \delta = 1, \ldots, 5 \tag{3.2}$$

wherein the Jacobian $\partial[g_1, \ldots, g_5]/\partial[u, \ldots, W] \not\equiv 0$. In §6 a powerful technique is provided for the derivation of complete sets for a given group; meanwhile it should be noted that η is formed from only the independent variables—i.e., η is an absolute invariant of the subgroup S. As will be seen, the importance of the absolute invariants lies in the fact that they become the similarity variables (i.e., the variables of the similarity representations).

Equations (3.2) suggest the reason two-parameter groups are invoked: With two-parameter groups the existence of similarity representations in a single independent variable η may be investigated. In [6] the analysis is further generalized by including *one*-parameter groups with the form (3.1) in the class C_G. Since the one-parameter groups have complete sets with the form $\{\eta_1(x, y, z),\ \eta_2(x, y, z),\ g_\delta(x, y, z, u, v, w, U, W)\}$, they are effective for investigating the existence of similarity representations in the two independent variables $\eta_1(x, y, z)$ and $\eta_2(x, y, z)$. However, for conciseness this discussion is not included here; see [6, §8].

4. The Invariance Analysis

Another elemental feature of all group approaches to similarity analysis is that the only groups considered are those which transform the problem "invariantly". Consequently, the first step of the formalism is to determine which—if any—of the members of C_G transform the problem at hand in this fashion. The invariance concept will now be developed; however a discussion of its importance and relationship to similarity analysis will be deferred until §5.

Insofar as the invariance analysis of partial differential equations is concerned, the procedure to be followed is modeled after one developed by Michal and Morgan. Thus, (2.1) is said to be transformed *invariantly* under (3.1) whenever

$$\overline{u_x}+\overline{v_y}+\overline{w_z}=H_1(a_1, a_2)[u_x+v_y+w_z] \tag{4.1}$$

for some function $H_1(a_1, a_2)$, which may be constant. The transformations in G, (3.1), are for the dependent and independent variables, and not for the derivatives. To transform the differential equations, transformations for the derivatives are obtained directly from G via chain rule operations: For example

$$\overline{u_x}=[C^u/C^x]u_x, \quad \overline{v_y}=[C^v/C^y]v_y, \quad \overline{w_{yy}}=[C^w/[C^y]^2]w_{yy}\,, \text{ etc.}$$

Thus, substitution into the left-side of (4.1) yields

$$[C^u/C^x]u_x+[C^v/C^y]v_y+[C^w/C^z]w_z=H_1(a_1, a_2)[u_x+v_y+w_z] \tag{4.2}$$

It follows, then, that (2.1) is transformed invariantly when

$$[C^u/C^x]=[C^v/C^y]=[C^w/C^z]\equiv H_1(a_1, a_2)\,. \tag{4.3}$$

In like manner, (2.2) is transformed invariantly under (3.1) whenever there is a function $H_2(a_1, a_2)$ such that

$$\begin{aligned}&\overline{uu_x}+\overline{vu_y}+\overline{wu_z}-\overline{vu_{yy}}-\overline{UU_x}-\overline{WU_z}\\&\qquad=H_2(a_1, a_2)\cdot[uu_x+vu_y+wu_z-vu_{yy}-UU_x-WU_z]\end{aligned} \tag{4.4}$$

Substitution into the left-side of (4.4) and rearrangement yields

$$\begin{aligned}&[[C^u]^2/C^x]uu_x+[C^vC^u/C^y]vu_y+[C^wC^u/C^z]wu_z-v[C^u/[C^y]^2]u_{yy}\\&-[[C^U]^2/C^x]UU_x-[C^WC^U/C^z]WU_z+R=H_2[uu_x+vu_y+wu_z-vu_{yy}-UU_x-W_z]\,,\end{aligned} \tag{4.5}$$

where

$$R\equiv\{[K^uC^u/C^x]u_x+[K^vC^u/C^y]u_y+[K^wC^u/C^z]u_z-[K^UC^U/C^x]U_x-[K^WC^U/C^z]U_z\}$$

Thus, it follows that (2.2) is transformed invariantly whenever

$$\begin{aligned}&\frac{[C^u]^2}{C^x}=\frac{C^uC^v}{C^y}=\frac{C^uC^w}{C^z}=\frac{C^u}{[C^y]^2}=\frac{[C^U]^2}{C^x}=\frac{C^UC^W}{C^z}\equiv H_2(a_1, a_2)\\&R\equiv 0:\quad K^u\equiv K^v\equiv K^w\equiv K^U\equiv K^W\equiv 0\end{aligned} \tag{4.6}$$

A like analysis shows that groups satisfying (3.1), (4.3) and (4.6) also transform (2.3) invariantly, and hence no further conditions arise. Therefore, (2.1, 2, 3) are *invariant in form* under such groups; i.e., when and only when (2.1, 2, 3) are satisfied,

$$\begin{aligned}&\overline{u_x}+\overline{v_y}+\overline{w_z}=0\\&\overline{uu_x}+\overline{vu_y}+\overline{wu_z}-\overline{vu_{yy}}-\overline{UU_x}-\overline{WU_z}=0\\&\overline{uw_x}+\overline{vw_y}+\overline{ww_z}-\overline{vw_{yy}}-\overline{UW_x}-\overline{WW_z}=0\end{aligned} \tag{4.7}$$

which have the same form as (2.1, 2, 3). Moreover, following Birkhoff, (2.4, 5, 6) are also *invariant in form* whenever the condition $K^y\equiv 0$ is appended to (4.3, 6); that is,

$$\begin{aligned}&\bar{u}(\bar{x}, 0, \bar{z})=\bar{v}(\bar{x}, 0, \bar{z})=\bar{w}(\bar{x}, 0, \bar{z})=0\\&\lim_{\bar{y}\to\infty}\bar{u}=\bar{U}\qquad\lim_{\bar{y}\to\infty}\bar{w}=\bar{W}\end{aligned} \tag{4.8}$$

[For example, since (4.6) yields $K^u \equiv 0$, (3.1) indicates $\bar{u}(\bar{x}, \bar{y}, \bar{z}) = C^u(a_1, a_2)u(x, y, z)$ where $\bar{y} \equiv C^y(a_1, a_2)y + K^y(a_1, a_2)$, etc.; thus, when $K^y \equiv 0$, $u(x, 0, z) = 0$ implies $\bar{u}(\bar{x}, 0, \bar{z}) = 0$.]

Summarizing, (2.1)–(2.6) are invariant under groups of the form G whenever (4.3, 6) are satisfied and $K^y \equiv 0$; that is, whenever the C's and K's satisfy

$$\begin{aligned} &C^u = C^U, \quad C^w = C^W, \quad C^v = [C^y]^{-1}, \quad C^x = C^u[C^y]^2, \quad C^z = C^w[C^y]^2 \\ &K^u = K^w = K^v = K^U = K^W = K^y \equiv 0 \end{aligned} \tag{4.9}$$

Thus, the foregoing restrictions indicate that groups which are of further interest are those in the class $C_{G'}$ with the form

$$G' \begin{cases} S' \begin{cases} \bar{x} = C^x(a_1, a_2)x + K^x(a_1, a_2) = C^u[C^y]^2 x + K^x(a_1, a_2) \\ \bar{y} = C^y(a_1, a_2)y \\ \bar{z} = C^z(a_1, a_2)z + K^z(a_1, a_2) = C^w[C^y]^2 z + K^z(a_1, a_2) \end{cases} \\ \bar{u} = C^u(a_1, a_2)u, \quad \bar{U} = C^u(a_1, a_2)U \\ \bar{v} = C^v(a_1, a_2)v = v/C^y(a_1, a_2) \\ \bar{w} = C^w(a_1, a_2)w, \quad \bar{W} = C^w(a_1, a_2)W \end{cases} \tag{4.10}$$

5. A Similarity Postulate

Comparison of (2.1)–(2.6) with (4.7, 8) reveals that the form is invariant under the transformations of any group G'. Following Birkhoff, this outcome suggests that solutions be sought which are also invariant in form under G'. That is, functions $\{I^u, I^v, I^w, I^U, I^W\}$ are to be sought such that when

$$\begin{aligned} &u = I^u(x, y, z), \quad v = I^v(x, y, z), \quad w = I^w(x, y, z) \\ &U = I^U(x, z), \quad W = I^W(x, z) \end{aligned} \tag{5.1}$$

then under G'

$$\begin{aligned} &\bar{u} = I^u(\bar{x}, \bar{y}, \bar{z}), \quad \bar{v} = I^v(\bar{x}, \bar{y}, \bar{z}), \quad \bar{w} = I^w(\bar{x}, \bar{y}, \bar{z}) \\ &\bar{U} = I^U(\bar{x}, \bar{z}), \quad \bar{W} = I^W(\bar{x}, \bar{z}) \end{aligned} \tag{5.2}$$

Solutions which exhibit this behavior are called *invariant solutions*, under G'. Indeed, it is invariant solutions which are obtained via similarity representations for (2.1)–(2.6). The procedure leading to invariant solutions is embodied in the following statement, which is called here for conciseness the *similarity postulate*:

> Whenever a given representation (such as (2.1)–(2.6)) in n ($n \geq 1$) dependent variables $(u_1, \ldots, u_n)$ and m ($m \geq 2$) independent variables $(x^1, \ldots, x^m)$ is *invariant* under an r-parameter group G_r of transformations (such as groups of the form G'), the problem can generally be rewritten in terms of $(n+m-r)$ variables. This is accomplished by a transformation of variables to a complete set of absolute invariants $(\eta_1, \ldots, \eta_{m-r}; g_1, \ldots, g_n)$ of G_r; the result is called a *similarity representation* of the problem. A solution $\{g_\delta = F_\delta(\eta_1, \ldots, n_{m-r}); \delta = 1, \ldots, \eta\}$ of the similarity representation defines an invariant solution $\{u_\delta = I_\delta(x^1, \ldots, x^m)\}$, implicitly:
>
> $$g_\delta(x^1, \ldots, x^m, I_1(x^1, \ldots, x^m), \ldots, I_n(\ldots)) = F_\delta(\eta_1(x^1, \ldots, x^m), \ldots, \eta_{m-r}(\ldots))$$
> $$(\delta = 1, \ldots, n)$$

A review of the literature indicates that Birkhoff was probably the first to propose such a procedure. Michal and Morgan have provided a rigorous basis for the method, but for representations consisting of differential equations *alone*—i.e., without auxiliary conditions. Finally it should be mentioned that the similarity postulate given here is a somewhat restricted version of one presented by the authors elsewhere; see [8], [9].

Returning to specific consideration of the problem given by (2.1)–(2.6) and the class of groups

(4.10), according to the similarity postulate there is only one major task which remains: the derivation of complete sets of absolute invariants. Indeed, deduction of the invariants leads simultaneously to nine distinct complete sets for the class of groups satisfying (4.10), which lead in turn to the nine cases of Table 1.

6. Complete Sets of Absolute Invariants

Heretofore it has been typical for absolute invariants to be established via inspection of, and/or trial with, the group; (e.g., see [7, pp. 12–13]). And, as a result of the simple forms of groups usually assumed at the outset, the lack of a systematic approach has been relatively unimportant. However, with the more complicated groups which can also arise, trial procedures to establish complete sets can be arduous, even when fruitful. Clearly then, a systematic technique for the derivation of complete sets is *desirable.*

The *key* feature of the systematic technique to be presented is the application of a basic theorem from group theory. To emphasize the essential features of the theorem in a relatively uncomplicated form, it is now quoted for the case of two-parameter groups S', (4.10).

Theorem:
A function $\eta(x, y, z)$ is an absolute invariant of a two-parameter group S': $\{\bar{x}=C^x(a_1, a_2)x+K^x(a_1, a_2), \bar{y}=C^y(a_1, a_2)y, \bar{z}=C^z(a_1, a_2)z+K^z(a_1, a_2)\}$ if and only if η satisfies the first order linear partial differential equations

$$\begin{aligned}
&[\alpha_1 x+\alpha_2]\frac{\partial\eta}{\partial x} + \alpha_3 y\frac{\partial\eta}{\partial y} + [\alpha_4 z+\alpha_5]\frac{\partial\eta}{\partial z} = 0\\
&[\beta_1 x+\beta_2]\frac{\partial\eta}{\partial x} + \beta_3 y\frac{\partial\eta}{\partial y} + [\beta_4 z+\beta_5]\frac{\partial\eta}{\partial z} = 0
\end{aligned} \tag{6.1}$$

where

$$\alpha_1 \equiv [\partial C^x/\partial a_1](a_1^\circ, a_2^\circ) = [[\partial C^u/\partial a_1]+2[\partial C^y/\partial a_1]](a_1^\circ, a_2^\circ),$$

$$\alpha_2 \equiv [\partial K^x/\partial a_1](a_1^\circ, a_2^\circ), \quad \alpha_3 \equiv [\partial C^y/\partial a_1](a_1^\circ, a_2^\circ),$$

$$\beta_1 \equiv [\partial C^x/\partial a_2](a_1^\circ, a_2^\circ) = [[\partial C^u/\partial a_2]+2[\partial C^y/\partial a_2]](a_1^\circ, a_2^\circ), \text{ etc.};$$

and wherein (a_1°, a_2°) denote the value of a_1 and a_2 which yield the identity: $\bar{x} = x$, $\bar{y} = y$ and $\bar{z} = z$.

By definition, for each of the two-parameter groups S' in the class $C_{G'}$ there is *one and only one* functionally independent solution to (6.1)—(the rank of the coefficient matrix for $\{\partial\eta/\partial x, \partial\eta/\partial y, \partial\eta/\partial z\}$ is *two*). Furthermore, if $\eta(x, y, z) \neq$ const. is a solution to (6.1), for a group S', then every other solution to (6.1), for S', is given in the form $H(\eta(x, y, z))$ where H is a differentiable function. It may be seen from (6.1) and the definitions of the constants α_i, β_i that differences between the groups S' are reflected by the α's and β's. That is, in general, any particular group S' possesses a characteristic set of α's and β's; and consequently a characteristic absolute invariant η is yielded by (6.1).

The extension of the foregoing theorem to a two-parameter group G' of the form (4.10) is straightforward and will be indicated in §7; for each such group the number of functionally independent absolute invariants g in a complete set equals the number of dependent variables—five for the problem at hand. (And for a *one*-parameter group S' there would be only a single differential equation of the form (6.1) to be satisfied; see [6, p. 16].) It should also be mentioned that the theorem is a specialization of a considerably more general result that has previously been applied by the authors for similarity analyses, [1], [6], [8], [9].

7. Derivation of Distinct Complete Sets

The similarity analysis of (2.1)–(2.6) now proceeds for the particular case of two-parameter groups of the form (4.10). The immediate objective is to establish a complete set for each such group; and as a *first* step toward this goal, attention is to be focused upon the problem of deriving the distinct η's, which are the independent variables of the similarity representations. The discussions of [6] indicate that each of the nine η's of Table 1 evolve *concurrently* when solutions to (6.1) are obtained via well-known standard techniques for solving linear partial differential equations (e.g., see [10, pp. 379–384]). However, for conciseness only two of the cases will be detailed here: Cases 1 and 2.

According to the theorem of § 6, for a two-parameter group S' there is one and only one solution to (6.1); that is, the coefficient matrix of $\{\partial\eta/\partial x, \partial\eta/\partial y, \partial\eta/\partial z\}$ must have rank two. The matrix has rank two whenever at least one of its two by two submatrices has a non-vanishing determinant; and this condition is met whenever *at least one* of the following is satisfied

$$\{[\lambda_{31}x+\lambda_{32}]\neq 0, \quad [\lambda_{34}z+\lambda_{35}]\neq 0, \quad [\lambda_{14}xz+\lambda_{15}x+\lambda_{24}z+\lambda_{25}]\neq 0 \tag{7.1}$$

wherein $\lambda_{ij}\equiv[\alpha_i\beta_j-\alpha_j\beta_i]$. For convenience, then, the system (6.1) will be rewritten in terms of the quantities given by (7.1); the result is

$$\begin{aligned} &[\lambda_{31}x+\lambda_{32}]\frac{\partial\eta}{\partial x}+[\lambda_{34}z+\lambda_{35}]\frac{\partial\eta}{\partial z}=0\\ &[\lambda_{31}x+\lambda_{32}]y\frac{\partial\eta}{\partial y}-[\lambda_{14}xz+\lambda_{15}x+\lambda_{24}z+\lambda_{25}]\frac{\partial\eta}{\partial z}=0\,. \end{aligned} \tag{7.2}$$

Thus, upon solving (7.2) in lieu of (6.1), differences between the groups S' are now reflected by the λ's. In particular, Cases 1 and 2 of Table 1 evolve for those groups S' for which $\lambda_{31}=\lambda_{34}=0$, $\lambda_{32}\neq 0$, $\lambda_{35}\neq 0$. According to [10, pp. 379–384], then, the first equation of (7.2) has the general solution

$$\eta=f(y,\ \xi(x,z)) \tag{7.3}$$

where

$$\xi(x,z)=\lambda_{35}x-\lambda_{32}z \tag{7.4}$$

However, to obtain a solution for the *system* (7.2) it is also necessary, of course, to satisfy the second equation as well as the first. Thus, with (7.3, 4), the second equation of (7.2) becomes

$$y\frac{\partial f}{\partial y}+[\lambda_{14}xz+\lambda_{15}x+\lambda_{24}z+\lambda_{25}]\frac{\partial f}{\partial\xi}=0 \tag{7.5}$$

since $\lambda_{31}=0$ and $\lambda_{32}\neq 0$.

The coefficient of $\partial f/\partial\xi$ in (7.5) is independent of y; thus, for f to be a function of y and ξ, it is clearly necessary for the coefficient to depend only upon ξ. That is, it is necessary for $\lambda_{14}=0$ and $\lambda_{32}\lambda_{15}=-\lambda_{24}\lambda_{35}$; then (7.5) becomes

$$y\frac{\partial f}{\partial y}+\left[\left[\frac{\lambda_{15}}{\lambda_{35}}\right]\xi+\lambda_{25}\right]\frac{\partial f}{\partial\xi}=0\,. \tag{7.6}$$

Upon invoking the above-mentioned standard technique once again, solutions to (7.6) are readily found to be in the form

$$f=\Phi(yH(\xi))\,, \tag{7.7}$$

where $H(\xi)$ is given via the ordinary differential equation

$$\left[\left[\frac{\lambda_{15}}{\lambda_{35}}\right]\xi+\lambda_{25}\right]\frac{d\ln H}{d\xi}=1 \tag{7.8}$$

obtained by substitution of (7.7) into (7.6).

Two distinct solutions may be obtained for (7.8), corresponding to (*i*) $\lambda_{15}\neq 0$ and (*ii*) $\lambda_{15}=0$. Thus, for $\lambda_{15}\neq 0$,

$$H(\xi) \sim \left[\left[\frac{\lambda_{15}}{\lambda_{35}}\right]\xi + \lambda_{25}\right]^{\lambda_{35}/\lambda_{15}} = [\lambda_{15}x + \lambda_{24}z + \lambda_{25}]^{\lambda_{35}/\lambda_{15}}. \tag{7.9}$$

Consequently, with (7.3, 7, 9) it follows that, for those groups S' with $\lambda_{14} = \lambda_{31} = \lambda_{34} = 0$, $\lambda_{32} \neq 0$, $\lambda_{35} \neq 0$, $\lambda_{15} \neq 0$, $\lambda_{32}\lambda_{15} = -\lambda_{24}\lambda_{35}$, absolute invariants are of the form

$$\eta^1 = \Phi^1(y[x + Az + B]^\mu) \tag{7.10}$$

which corresponds to Case 1 of Table 1.

Similarly, for $\lambda_{15} = 0$, (7.8) yields the solution

$$H(\xi) \sim \exp(\xi/\lambda_{25}) = \exp(\lambda_{35}x/\lambda_{25})\exp(-\lambda_{32}z/\lambda_{25}) \tag{7.11}$$

Thus, with (7.3, 7, 11) it follows that, for those groups S' with $\lambda_{14} = \lambda_{15} = \lambda_{24} = \lambda_{31} = \lambda_{34} = 0$, $\lambda_{32} \neq 0$, $\lambda_{35} \neq 0$, absolute invariants are of the form

$$\eta^2 = \Phi^2(y \exp(\mu x) \exp(mz)), \tag{7.12}$$

which corresponds to Case 2 of Table 1.

In like manner, additional distinct η's, corresponding to cases of Table 1, may be obtained from (7.2). Thus, Case 3 evolves when $\lambda_{31} \neq 0$, $\lambda_{34} \neq 0$, Case 4 is obtained when $\lambda_{31} = 0$, $\lambda_{34} \neq 0$, $\lambda_{32} \neq 0$, $\lambda_{35} \neq 0$, absolute invariants are of the form

A complete set for a group G' not only includes an $\eta(x, y, z)$ but also five functionally independent g's—vid. (3.2). The procedure to be followed in deriving the g's is parallel to that used in obtaining the η's. Thus, extending the theorem of §6 to the groups G', five independent solutions $g(x, y, z, u, v, w, U, W)$ are to be established for

$$\begin{aligned} &\alpha_6 u \frac{\partial g}{\partial u} - \alpha_3 v \frac{\partial g}{\partial v} + \alpha_7 w \frac{\partial g}{\partial w} + \alpha_6 U \frac{\partial g}{\partial U} + \alpha_7 W \frac{\partial g}{\partial W} + \\ &\qquad + [\alpha_1 x + \alpha_2]\frac{\partial g}{\partial x} + \alpha_3 y \frac{\partial g}{\partial y} + [\alpha_4 z + \alpha_5]\frac{\partial g}{\partial z} = 0 \\ &\beta_6 u \frac{\partial g}{\partial u} - \beta_3 v \frac{\partial g}{\partial v} + \beta_7 w \frac{\partial g}{\partial w} + \beta_6 U \frac{\partial g}{\partial U} + \beta_7 W \frac{\partial g}{\partial W} + \\ &\qquad + [\beta_1 x + \beta_2]\frac{\partial g}{\partial x} + \beta_3 y \frac{\partial g}{\partial y} + [\beta_4 z + \beta_5]\frac{\partial g}{\partial z} = 0 \end{aligned} \tag{7.13}$$

wherein $\alpha_6 \equiv [\partial C^u/\partial a_1](a_1^\circ, a_2^\circ) = \alpha_1 - 2\alpha_3$, $\beta_6 \equiv [\partial C^u/\partial a_2](a_1^\circ, a_2^\circ) = \beta_1 - 2\beta_3$, etc.

Three functionally independent g's may be readily obtained as solutions to (7.13) by inspection. Thus, for every group G' of $C_{G'}$,

$$\left\{\begin{aligned} g_1(u, U) &= \Gamma_1(u/U) \\ g_2(v, y) &= \Gamma_2(vy) \\ g_3(w, W) &= \Gamma_3(w/W) \end{aligned}\right\} \tag{7.14}$$

where the Γ's are arbitrary differentiable functions.

To obtain complete sets, then, it only remains to determine two additional independent g's for each group. Perhaps the most direct procedure to follow in accomplishing this objective is to determine solutions to (7.13) with the forms: $g_4(x, U, z)$, $g_5(x, W, z)$, for then the same sequence of steps may be followed as in the derivation of the invariants $\eta(x, y, z)$ given by (7.10, 12). In analogy to the previous analysis of (6.1) for the η's, but with U and W playing (in turn) the role of y, two additional independent g's may be obtained with the forms

$$\begin{aligned} g_4(x, U, z) &= \Gamma_4(U\hat{\omega}_4(x, z)) \\ g_5(x, W, z) &= \Gamma_5(W\hat{\omega}_5(x, z)) \end{aligned} \tag{7.15}$$

Specific expressions for $\hat{\omega}_4$ and $\hat{\omega}_5$ could be deduced at this point using procedures analogous to those employed previously to obtain the arguments of (7.10, 12)—for details, see [6, pp. 48, 49]. However, as will be shown in the following, for the present application it is possible to simply forego this step. Thus, one merely proceeds with g_4 and g_5 in the forms of (7.15), and it is found that the similarity representations can be established without foreknowledge of $\hat{\omega}_4$ and $\hat{\omega}_5$; and forms for $\hat{\omega}_4$ and $\hat{\omega}_5$ evolve naturally as a part of the development.

8. Development of the Similarity Representations

Thusly, for the cases being detailed here, forms for the invariants, needed to obtain similarity representations, have been established. According to the similarity postulate, therefore, similarity representations should be sought via changes of variables: $F_\delta(\eta)=g_\delta$. To illustrate this procedure simply, the Φ's and Γ's of (7.10, 12, 14, 15) are each selected to be the identity function. That is, the particular cases given by $\{\eta=y\pi(x, z)$ where $\pi_1=[x+Az+B]^\mu$ or $\pi_2=\exp(\mu x)$ $\exp(mz)$, $g_1=u/U$, $g_2=vy$, $g_3=w/W$, $g_4=U\hat{\omega}_4(x, z)$, $g_5=W\hat{\omega}_5(x, z)\}$ will be considered.

For the cases under consideration, then, similarity representations are sought via transformations $\{F_1(\eta)=u/U$, $F_2(\eta)=vy$, $F_3(\eta)=w/W$, $F_4(\eta)=U\hat{\omega}_4(x, z)$, $F_5(\eta)=W\hat{\omega}_5(x, z)\}$. As a first step it should be noted that F_4 must equal a constant, U_0, since U and $\hat{\omega}_4$ are independent of y whereas η is not. Similarly, F_5 equals a constant, W_0. So,

$$U(x, z) = U_0\omega_4(x, z) \tag{8.1}$$
$$W(x, z) = W_0\omega_5(x, z) \tag{8.2}$$

where, for convenience, $\omega_4\equiv[\hat{\omega}_4]^{-1}$ and $\omega_5\equiv[\hat{\omega}_5]^{-1}$.

The remainder of the analysis is straightforward. Thus, with $\eta=y\pi$, $u=U_0\omega_4F_1(\eta)$, $v=F_2(\eta)/y$, $w=W_0\omega_5F_3(\eta)$ and elementary chain rule operations (2.1), for example, may be rewritten as

$$\begin{aligned}\left[\frac{U_0}{\pi^2}\frac{\partial\omega_4}{\partial x}\right]\eta^2F_1 + \left[U_0\frac{\omega_4}{\pi^3}\frac{\partial\pi}{\partial x}\right]\eta^3\frac{dF_1}{\partial\eta} + \left[\frac{W_0}{\partial\eta}\frac{\partial\omega_5}{\partial z}\right]\eta^2F_3 \\ + \left[W_0\frac{\omega_5}{\pi^3}\frac{\partial\pi}{\partial z}\right]\eta^3\frac{dF_3}{d\eta} + \eta\frac{dF_2}{d\eta} - F_2 = 0\,.\end{aligned} \tag{8.3}$$

Inasmuch as the last term of (8.3) has a constant coefficient, for (8.3) to reduce to an expression in the single independent variable η as indicated by the similarity postulate, it is necessary that the remaining coefficients be functions of η alone. Thus, since π, ω_4, ω_5 are independent of y,

$$[U_0\omega_4/\pi^3][\partial\pi/\partial x]\equiv C_1,\ [W_0\omega_5/\pi^3][\partial\pi/\partial z]\equiv C_2 \tag{8.4}$$

$$[U_0/\pi^2][\partial\omega_4/\partial x]\equiv C_3,\ [W_0/\pi^2][\partial\omega_5/\partial z]\equiv C_4\,, \tag{8.5}$$

where the C's are constants. With (8.3, 4, 5) a similarity representation for (2.1) may be obtained for each of the cases—namely, (2.7). Similar steps lead to (2.8)–(2.11) and the constants C_5, C_6,

$$[W_0\omega_5/\omega_4\pi^2][\partial\omega_4/\partial z]\equiv C_5,\ [U_0\omega_4/\omega_5\pi^2][\partial\omega_5/\partial x]\equiv C_6\,. \tag{8.6}$$

The only remaining tasks are to utilize each of the η's, $\eta=y\pi_i$ $(i=1, 2)$, in turn with (8.4)–(8.6) to (*i*) evaluate the C's appearing in the *corresponding* similarity representation, and (*ii*) to evaluate the *corresponding* expressions for ω_4 and ω_5. To illustrate these steps, Case 1 with $\eta=y\pi_1(x, z)$ $=y[x+Az+B]^\mu$, will now be considered: Since $\pi_1=[x+Az+B]^\mu$, (8.4) yield respectively,

$$[\mu U_0/C_1]\omega_4=[A\mu W_0/C_2]\omega_5=[x+Az+B]^{2\mu+1} \tag{8.7}$$

For convenience, let $C_1=\mu U_0$ and $C_2=A\mu W_0$; then with (8.1, 2) and (8.7), it follows that

$$\begin{aligned}U/U_0=\omega_4=[x+Az+B]^{2\mu+1}\\ W/W_0=\omega_5=[x+Az+B]^{2\mu+1}\end{aligned} \tag{8.8}$$

In like manner with π_1 and (8.8), equations (8.5) and (8.6) yield

$$\begin{aligned}[C_3/U_0] &= [C_6/U_0] = [2\mu+1] \\ [C_4/W_0] &= [C_5/W_0] = A[2\mu+1]\end{aligned} \tag{8.9}$$

The foregoing conclusions regarding the C's and ω's are reported in Table 1 as Case 1. Analogously, the entries corresponding to Case 2 may be derived via the above procedure when $\pi_2 = \exp(\mu x)\exp(mz)$ is invoked. The remaining cases of Table 1 may be obtained in a like manner.

9. Closure

In this paper, a new systematic formalism for similarity analyses has been introduced. To summarize, the systematic formalism reported herein has three principal steps, each of which include advantages over the usual group approach: (1) An analysis is initiated with a general class of groups, (3.1), rather than by means of the specialized choices with limited capacity that have traditionally been the starting point. (2) Then, upon invoking the invariance concept, an appropriate subclass is deduced; the deduction procedure considers not only the differential equations but also the auxiliary conditions. (3) Finally, complete sets of absolute invariants—the similarity variables—are derived in a systematic manner, rather than by the traditional trial and/or inspection procedures.

Compared to the group methods for similarity analysis presented earlier by the authors [1], the present formalism has a number of desirable features: (*i*) The formalism explicitly considers the case of r-parameter groups, in contrast to the limited one-parameter discussion of [1]. (*ii*) When an analysis is initiated with a class of groups exhibiting the form (3.1) rather than the form utilized in [1], subsequent manipulations are significantly simplified—in particular, those required for the deduction of the subclass under which the problem transforms invariantly. On the other hand, as noted previously, groups of the form (3.1) are found to be very satisfactory for an extensive realm of practical problems. (*iii*) Too, initiating an analysis with the groups (3.1) assures that absolute invariants (the similarity variables) may be very readily deduced via the straightforward procedures described in §6, 7. (*iv*) As illustrated by the present discussion, it may not be necessary to obtain detailed expressions for each of the invariants of a complete set, in order to obtain similarity representations. Thus, for the case at hand, it was possible to forego determination of expressions for $\omega_4(x, z)$ and $\omega_5(x, z)$ until after the system of equations (2.7)–(2.11) for the similarity representations was developed; consequently, considerable effort was saved. Another advantage of foregoing such details until after the similarity system is developed is the following. (*v*) For all the cases which evolve simultaneously, a single system of equations arises to represent the original system—here, (2.7)–(2.11) arose to represent (2.1)–(2.6), for each of the cases of Table 1. And this similarity system is derived in one step for all of the different cases. (*vi*) Finally, while the present discussion focuses attention only upon the problem of deducing similarity representations for (2.1)–(2.6) in a single independent variable $\eta(x, y, z)$, it is also possible to utilize the *same* analysis to establish similarity representations in two independent variables; see [6, §8]. And such representations have significance inasmuch as they typically admit more general forms for the unspecified functions appearing in a given representation—here the free-stream velocity components $U(x, z)$ and $W(x, z)$—than are obtainable via two-parameter groups.

The formalism introduced here is well suited for the similarity analyses of practical problems. Thus, the formalism has yielded (2.7)–(2.11), which constitute *nine* distinct similarity representations for the given system (2.1)–(2.6), without the need for explicitly introducing specific groups of transformations as in previous group methods [7, pp. 10–16], or to introduce *ad hoc* assumptions concerning the form of the solution as in previous non-group methods [5, pp. 4–6]. Moreover, not only does the formalism presented here systematize similarity analyses, it also may yield more general conclusions than the heretofore conventional methods. That is, as indicated in the foregoing presentation the cases reported in Table 1 evolve *concurrently* via a single systematic approach; and as may be established by comparison of Table 1 with [5, pp. 28–30] the similarity variables reported here are in every instance *as least* as general, and in some instances are more general in form. It is believed, therefore, that the systematic formalism introduced here significantly facilitates similarity analyses of partial differential equations with auxiliary conditions.

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